

GOLDMAN BRACKET & LENGTH EQUIVALENT CURVES

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ABSTRACT. A pair of distinct free homotopy classes of closed curves in an orientable surface F with negative Euler characteristic is said to be length equivalent if for any hyperbolic structure on F , the length of the geodesic representative of one class is equal to the length of the geodesic representative of the other class. Suppose α and β are two intersecting oriented closed curves on F and P and Q are any two intersection points between them. If the two terms $\langle \alpha *_P \beta \rangle$ and $\langle \alpha *_Q \beta \rangle$ in $[\langle \alpha \rangle, \langle \beta \rangle]$, the Goldman bracket between them, are the same, then we construct infinitely many pairs of length equivalent curves in F . These pairs correspond to the terms of the Goldman bracket between a power of α and β . As a special case, our construction shows that given a self-intersecting geodesic α of F and any self-intersection point P of α , we get a sequence of such pairs.

1. INTRODUCTION

Let F be an orientable surface with negative Euler characteristic. By [9, Theorem 1.2], we can endow F with a hyperbolic metric. By a *hyperbolic surface* we mean an orientable surface of finite type with negative Euler characteristic and with a given hyperbolic metric.

Given a hyperbolic surface F , there is a bijective correspondence between the set of non-trivial free homotopy classes of oriented closed curves and the set of non-trivial conjugacy classes of $\pi_1(F)$ [4, Theorem 2.6]. Also in every non-trivial, non-peripheral free homotopy class of a closed curve, there exists a unique geodesic [9, Proposition 1.3]. Henceforth we use free homotopy classes of oriented closed curves and conjugacy classes of elements in $\pi_1(F)$ interchangeably. Given any oriented closed curve β , we denote both its free homotopy class in F and its conjugacy class in $\pi_1(F)$ by $\langle \beta \rangle$.

Two distinct conjugacy classes $\langle \alpha \rangle$ and $\langle \beta \rangle$ in $\pi_1(F)$ are said to be *length equivalent* if for every hyperbolic metric in F , the length of the unique geodesic representative of the free homotopy class $\langle \alpha \rangle$ is equal

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to the length of the unique geodesic representative of the free homotopy class $\langle \beta \rangle$ and $\langle \beta \rangle \neq \langle \alpha^{-1} \rangle$.

Given two free homotopy classes of oriented closed curves $\langle \alpha \rangle$ and $\langle \beta \rangle$ in F , consider two oriented closed curves α and β representing $\langle \alpha \rangle$ and $\langle \beta \rangle$ respectively. Performing a small homotopy if necessary, we assume that α and β intersect transversally in double points. Goldman [10] defined the bracket of $\langle \alpha \rangle$ and $\langle \beta \rangle$ as the sum,

$$[\langle \alpha \rangle, \langle \beta \rangle] = \sum_{P \in \alpha \cap \beta} \epsilon_P \langle \alpha *_P \beta \rangle$$

where $\alpha \cap \beta$ denotes the set of all intersection points between α and β , ϵ_P denotes the sign of the intersection between α and β at P and $(\alpha *_P \beta)$ denotes the loop product of α and β at P .

Main Theorem. *Let F be a hyperbolic surface and α and β be two closed oriented curves intersecting transversally in double points. Also assume that they intersect each other minimally in their free homotopy class. Let P_1 and P_2 be two intersection points between them such that $\langle \alpha *_P \beta \rangle = \langle \alpha *_P \beta \rangle$, i.e. the terms corresponding to P_1 and P_2 in $[\langle \alpha \rangle, \langle \beta \rangle]$ are same. Then there exists $g, h \in \pi_1(F)$ and a positive integer N such that for all $n > N$, the conjugacy classes of $\alpha^n \beta^g$ and $\alpha^n \beta^h$ are length equivalent.*

We first consider the following special case. Let α be any self-intersecting geodesic and P be any self-intersection point of α . As the Goldman bracket is a Lie bracket, $[\langle \alpha \rangle, \langle \alpha \rangle] = 0$. The reason is the following. Let α_1 and α_2 be two curves freely homotopic to α which intersect each other transversally and $\alpha_1 \cup \alpha_2$ cobounds a narrow annulus. Then corresponding to each self intersection point P of α , we get two intersection points P_1 and P_2 between α_1 and α_2 and $\langle \alpha_1 *_P \beta \rangle = \langle \alpha_1 *_P \beta \rangle$ with opposite signs. Now we are in the same situation as the above theorem and we have the following result.

Theorem 1.1. *Let F be any hyperbolic surface and α be any self-intersecting closed geodesic in F . Given any self intersection point P of α , there exists $g \in \pi_1(F)$ and a positive integer N such that for all $n > N$, the conjugacy classes of $\alpha^n g \alpha g^{-1}$ and $g \alpha^n g^{-1} \alpha$ are length equivalent.*

We first prove the Theorem 1.1. Then we prove the Main Theorem which follows from the proof of Theorem 1.1.

Length equivalent curves. Given any hyperbolic surface F , the *length spectrum* of F is the set of the lengths of all geodesics in F . It is well known that the length spectrum of a hyperbolic surface is

discrete [3, Theorem 1.6.11]. Randol [14], using a result of Horowitz [11], proved that on every hyperbolic surface, the length spectrum has arbitrarily high multiplicities. Buser [3, Section 3.7], gave an algorithm to construct length equivalent curves in the pairs of pants and in torus with one boundary component. Chas [8] provided examples of length equivalent curves with different self-intersection number. For more details about length equivalent curves, see [1].

Goldman bracket. William Goldman [10], discovered the Goldman bracket in the eighties and proved that if the Goldman bracket between two curves is zero and one of them is simple then they have disjoint representatives. Chas [6],[7], proved a stronger version of Goldman's result namely if one of the curves has a simple representative then the number of terms in the Goldman bracket is the same as their geometric intersection number. Chas and Krongold [5], proved that for a compact surface with non-empty boundary, the Lie bracket between an element and its cube determines the self-intersection number. Using hyperbolic geometry, Chas and Gadgil [4], proved that there exists a positive integer m such that for all $n > m$, the geometric intersection number between x and y is the number of terms in the Goldman bracket between n -th power of x and y divided by n . For more details about Goldman bracket, see [4], [10].

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2. NOTATION AND DEFINITION

Let F be an orientable surface of negative Euler characteristic, i.e. F is an orientable surface of genus g with b boundary components and n punctures such that $2 - 2g - b - n < 0$.

Fix a hyperbolic metric on F . We identify the fundamental group $\pi_1(F)$ of F with a discrete subgroup of $PSL_2(\mathbb{R})$, the group of orientation preserving isometries of the upper half plane \mathbb{H} . The action of $\pi_1(F)$ on \mathbb{H} is properly discontinuous and without any fixed point and the quotient space is isometric to F . Henceforth by an isometry of \mathbb{H} we mean an orientation preserving isometry. Given α and g in $\pi_1(F)$, we denote $g\alpha g^{-1}$ by α^g and the translation length of α by τ_α . If α is hyperbolic then α^g is also hyperbolic with $\tau_{\alpha^g} = \tau_\alpha$ and $A_{\alpha^g} = gA_\alpha$ for all $g \in \pi_1(F)$ where A_α denotes the axis of α .

By a *lift* of a closed curve γ to \mathbb{H} we mean the image of a lift $\mathbb{R} \rightarrow \mathbb{H}$ of the map $\gamma \circ \pi$ where $\pi : \mathbb{R} \rightarrow S^1$ is the usual covering map.

Let $\langle \alpha \rangle$ be the free homotopy class of an oriented closed self-intersecting geodesic α . Corresponding to each self intersection point P of α and each positive integer n , we construct two free homotopy classes of oriented closed curves in the following way.

Consider a narrow annular neighbourhood of α . Let α_1 (respectively α_2) denote the boundary curve of the annulus at the right hand side (respectively left hand side) of α (Figure 1). The curves α_1 and α_2 intersect each other transversally and $\alpha_1 \cup \alpha_2$ cobounds a narrow annulus. Therefore corresponding to each self intersection point P of α , we get two intersection points P_1 and P_2 between α_1 and α_2 .

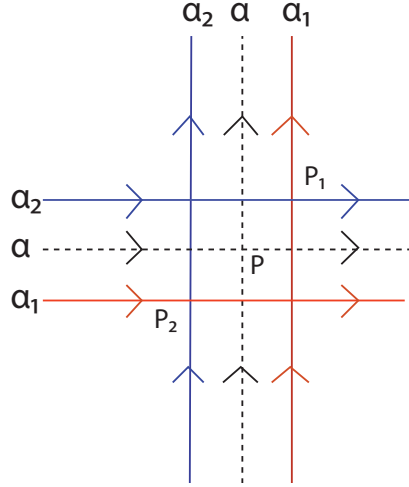


FIGURE 1. Corresponding to the self-intersection point P of α , there are two intersection points P_1 and P_2 between α_1 and α_2 . The sign of the intersection between α_1 and α_2 at P_1 and P_2 respectively are opposite to each other.

Let $\Gamma_1(P, n)$ be the loop product of α_1^n and α_2 at the point P_1 and $\Gamma_2(P, n)$ be the loop product of α_1^n and α_2 at the point P_2 , where γ^n denotes the curve which goes around the curves γ , n -times. Observe that the free homotopy class of γ^n is well defined.

Let ϵ_i denote the sign of the intersection between α_1 and α_2 at P_i for $i = 1, 2$. From Figure 1, we get

$$\epsilon_1 = -\epsilon_2.$$

In the next section we construct lifts of a curve from the free homotopy class $\langle \Gamma_i(P, n) \rangle$ to show that the free homotopy class $\langle \Gamma_i(P, n) \rangle$

depends only on the geodesic α , the self intersection point P and the integer n (Lemma 3.1).

We prove the following lemma which is the main lemma of this paper. Theorem 1.1 follows at once from this lemma.

Lemma 2.1. *There exists a positive integer N such that for all $n > N$, the free homotopy classes $\langle \Gamma_1(P, n) \rangle$ and $\langle \Gamma_2(P, n) \rangle$ are length equivalent.*

Remark 1. There is a geometric way to see that the geodesics in the free homotopy classes of $\langle \Gamma_1(P, n) \rangle$ and $\langle \Gamma_2(P, n) \rangle$ are of equal length.

First observe that any non-simple closed geodesic α is the immersion of a figure eight geodesic. Furthermore, you can choose any self-intersection point of α to be the image of the self-intersection point of the figure eight geodesic. This immersion extends to an immersed pair of pants (for instance by taking a ϵ -neighbourhood of the immersed curve). Denote by Y the lift of the immersed pants by the immersion. Now it is easy to see that the lifts of $\langle \Gamma_1(P, n) \rangle$ and $\langle \Gamma_2(P, n) \rangle$ to the pair of pants Y are reflected images of one another on Y and thus have the same length. Showing they are not conjugate to each other in $\pi_1(F)$ is not straightforward.

3. LIFTS OF THE CURVES

In this section we construct a lift of a piecewise geodesic curve in the free homotopy class $\langle \Gamma_i(P, n) \rangle$ and describe the unique geodesic in the free homotopy class $\langle \Gamma_i(P, n) \rangle$.

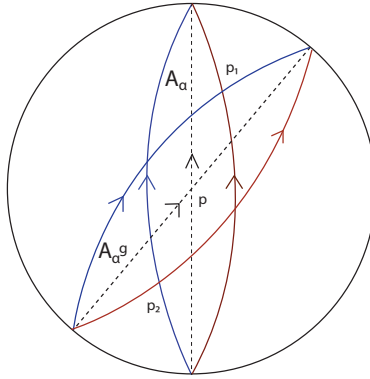


FIGURE 2. Lift of the Figure 1 to the unit disc.

Let α be a self-intersecting geodesic in an oriented hyperbolic surface F . Let P be a self-intersection point of α . We consider P to be the

base-point of the fundamental group of F . Therefore we consider α to be an element of $\pi_1(F)$.

Let A_α be the axis of α in the upper half plane \mathbb{H} . Let p be a lift of the self-intersection point P on A_α . By [4, Section-3], there exists g in $\pi_1(F)$ such that $gA_\alpha \cap A_\alpha = \{p\}$. The axis gA_α is same as the axis of α^g .

From Figure 2, the sign of the intersection between A_α and A_{α^g} at p is ϵ_1 and the sign of the intersection between A_{α^g} and A_α at p is ϵ_2 .

Let $\beta_1 : [0, 1] \rightarrow \mathbb{H}$ be a curve from $\beta_1(0) = (\alpha^g)^{-1}p$ to $\beta_1(1) = \alpha^n p$, whose image is given by the concatenation of the geodesic segment of A_{α^g} from $(\alpha^g)^{-1}p$ to p with the geodesic segment of A_α from p to $\alpha^n p$. As $\alpha^n \alpha^g(\beta_1(0)) = \beta_1(1)$, we extend β_1 periodically to a map $\gamma_1(p, n) : \mathbb{R} \rightarrow \mathbb{H}$ such that $\gamma_1(p, n)(t) = \beta_1(t)$ for $t \in [0, 1]$ and $\gamma_1(p, n)(t+1) = \alpha^n \alpha^g(\gamma_1(p, n)(t))$ for all t .

Similarly let $\beta_2 : [0, 1] \rightarrow \mathbb{H}$ be a curve from $\beta_2(0) = (\alpha)^{-1}p$ to $\beta_2(1) = (\alpha^g)^n p$, whose image is given by the concatenation of the geodesic segment of A_α from $(\alpha)^{-1}p$ to p with the geodesic segment of A_{α^g} from p to $(\alpha^g)^n p$. As $(\alpha^g)^n \alpha(\beta_2(0)) = \beta_2(1)$, we extend β_2 periodically to a map $\gamma_2(p, n) : \mathbb{R} \rightarrow \mathbb{H}$ such that $\gamma_2(p, n)(t) = \beta_2(t)$ for $t \in [0, 1]$ and $\gamma_2(p, n)(t+1) = (\alpha^g)^n \alpha(\gamma_2(p, n)(t))$ for all t .

We call the lifts of P in $\gamma_i(p, n)$ as the *vertices* of $\gamma_i(p, n)$.

Remark 2. As α is freely homotopic to α_1 and α_2 , the projection of β_1 is freely homotopic to $\Gamma_1(P, n)$ and the projection of β_2 is freely homotopic to $\Gamma_2(P, n)$.

Remark 3. Let $\mu_1 = \alpha^n \alpha^g$ and $\mu_2 = (\alpha^g)^n \alpha$. Denote the geodesic arc from P to $\alpha^n(P)$ by I_1 , the geodesic segment from $\alpha^n(P)$ to $\alpha^n \alpha^g(P)$ by I_2 , the geodesic segment from P to $(\alpha^g)^n$ by J_1 and the geodesic segment from $(\alpha^g)^n(P)$ to $(\alpha^g)^n \alpha(P)$ by J_2 . Then $\gamma_1(p, n)$ consists of geodesic segments of the form $\mu_1^k(I_1)$ and $\mu_2^k(I_2)$ occurring alternately. Similarly $\gamma_2(p, n)$ consists of geodesic segments of the form $\mu_1^k(J_1)$ and $\mu_2^k(J_2)$ occurring alternately.

Every vertex of $\gamma_1(p, n)$ corresponds to an intersection point between $\mu_1^k(I_1)$ and $\mu_2^k(I_2)$. Similarly, every vertex of $\gamma_2(p, n)$ corresponds to an intersection point between $\mu_1^k(J_1)$ and $\mu_2^k(J_2)$.

Remark 4. The sign of the intersection between $\mu_1^k(I_1)$ and $\mu_2^k(I_2)$ at any vertex of $\gamma_1(p, n)$ is equal to ϵ_1 . Similarly, the sign of the intersection between $\mu_1^k(J_1)$ and $\mu_2^k(J_2)$ at any vertex of $\gamma_2(p, n)$, is equal to ϵ_2 .

By [4, Lemma 7.1], $\gamma_i(p, n)$ is a quasi-geodesic for $i \in \{1, 2\}$. As every quasi geodesic in \mathbb{H} is homotopic (fixing endpoints) to a unique

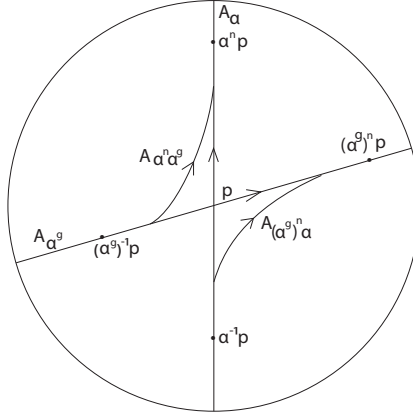


FIGURE 3.

geodesic, by Remark 2, the unique geodesic corresponding to the free homotopy class $\langle \Gamma_i(P, n) \rangle$ is the projection of the axis of the quasi geodesic $\gamma_i(p, n)$.

By [2, Theorem 7.38.6] and Figure 3, the axis of $\gamma_1(p, n)$ is the axis of $\alpha^n \alpha^g$ and the axis of $\gamma_2(p, n)$ is the axis of $(\alpha^g)^n \alpha$.

Therefore the free homotopy class $\langle \Gamma_1(P, n) \rangle$ corresponds to the conjugacy class of $\alpha^n \alpha^g$ and the free homotopy class $\langle \Gamma_2(P, n) \rangle$ corresponds to the conjugacy class of $(\alpha^g)^n \alpha$.

From the above discussion we obtain the following lemma.

Lemma 3.1. *Given any self-intersecting geodesic α in F , any self-intersection point P of α and any positive integer n , the two free homotopy classes $\langle \Gamma_1(P, n) \rangle$ and $\langle \Gamma_2(P, n) \rangle$ correspond to the conjugacy classes of $\alpha^n \alpha^g$ and $(\alpha^g)^n \alpha$ respectively, where $g \in \pi_1(F)$ is determined by a fixed lift p of P on A_α .*

4. FINAL RESULTS

Lemma 4.1. *Let α and g be two elements in $\pi_1(F)$ such that A_α intersects A_{α^g} . Then for any hyperbolic metric X on F , $l_X(\alpha^n \alpha^g) = l_X((\alpha^g)^n \alpha)$, where $l_X(\alpha)$ denotes the length of the geodesic freely homotopic to α in the metric X .*

Proof. We give two proofs of this lemma. The first one is geometric and the second one is algebraic.

Geometric proof: Let X be any hyperbolic metric in F and θ be the angle between A_α and A_{α^g} at the point P in the positive direction of both axes. Consider the triangles $\triangle pqr$ and $\triangle pq_1r_1$ in Figure 4 and the angle $\mu = \pi - \theta$.

By [2, Theorem 7.38.6], we have the following:

- $l(pq) = l_X(\alpha^n)/2 = n\tau_\alpha/2 = n\tau_{\alpha^g}/2 = l_X((\alpha^g)^n)/2 = l(pq_1)$.
- $l(rp) = l_X(\alpha^g)/2 = \tau_{\alpha^g}/2 = \tau_\alpha/2 = l_X(\alpha)/2 = l(r_1p)$.
- $\angle rpq = \mu = \angle r_1pq_1$.
- $l(rq) = l_X(\alpha^n\alpha^g)/2$ and $l(r_1q_1) = l_X((\alpha^g)^n\alpha)/2$.

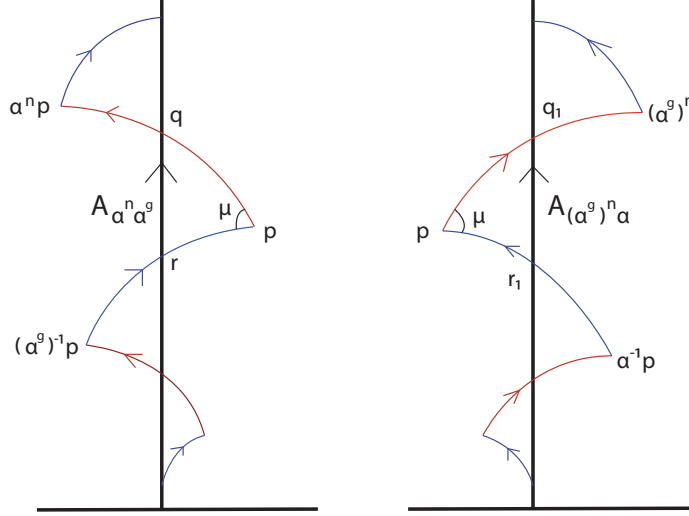


FIGURE 4.

Therefore by the hyperbolic cosine rule,

$$l_X(\alpha^n\alpha^g) = l_X((\alpha^g)^n\alpha).$$

Algebraic proof: We use the following equalities of $PSL_2(\mathbb{R})$ which follows from straightforward computation. For details see [3, Lemma 3.7.9]

If A and B belong to $PSL_2(\mathbb{R})$ then we have the following equalities.

$$\text{trace}(A) = \text{trace}(A^{-1})$$

$$\text{trace}(AB) = \text{trace}(BA)$$

$$\text{trace}(A) = \text{trace}(BAB^{-1})$$

$$\text{trace}(AB) = \text{trace}(A)\text{trace}(B) - \text{trace}(AB^{-1})$$

If A is a matrix representing the geodesic α in X then

$$l_X(\alpha) = \cosh^{-1}\left(\frac{|\text{trace}(A)|}{2}\right).$$

Let $A = \alpha$ and $B = \alpha^g$. By the above comment, it is sufficient to show that $\text{trace}(A^n B) = \text{trace}(B^n A)$ for all n in \mathbb{N} . We prove this using induction on n .

For $n = 1$, this follows from the second equation.

Suppose the result is true for all $k < n$. Using the above equations, we have

$$\begin{aligned}
 \text{trace}(A^n B) &= \text{trace}(BA^n). \\
 &= \text{trace}(BA^{n-1})\text{trace}(A) - \text{trace}(BA^{n-2}). \\
 &= \text{trace}(A^{n-1}B)\text{trace}(A) - \text{trace}(A^{n-2}B). \\
 &= \text{trace}(B^{n-1}A)\text{trace}(B) - \text{trace}(B^{n-2}A) \\
 &= \text{trace}(AB^{n-1})\text{trace}(B) - \text{trace}(AB^{n-2}) \\
 &= \text{trace}(AB^n) \\
 &= \text{trace}(B^n A).
 \end{aligned}$$

□

Lemma 4.2. *There exists a positive integer $N > 1$ such that for all $n > N$, $\alpha^n \alpha^g$ is not conjugate to $(\alpha^g)^n \alpha$ or $((\alpha^g)^n \alpha)^{-1}$.*

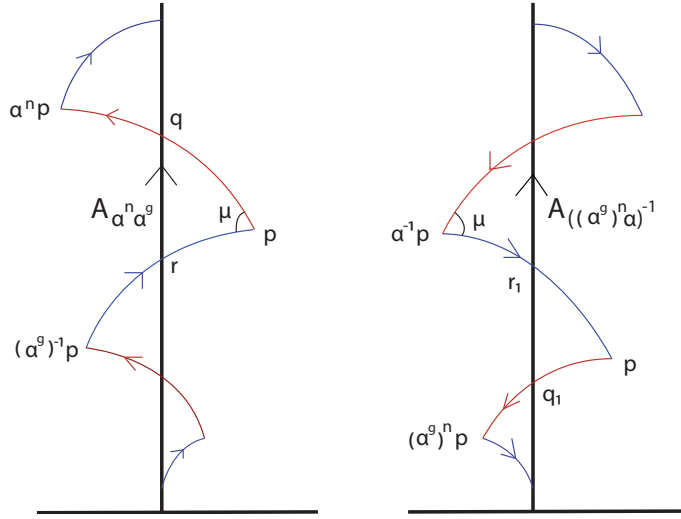


FIGURE 5.

Proof. Conjugacy of two elements in $\pi_1(F)$ is independent of the hyperbolic metric in F . So we fix a metric and show that there exists an integer $N > 1$ such that for all $n > N$, $\alpha^n \alpha^g$ is not conjugate to $(\alpha^g)^n \alpha$ or $((\alpha^g)^n \alpha)^{-1}$. Then this is true for all hyperbolic metric in F .

Fix a hyperbolic metric X on F . Let $l_X(\alpha) = L$. Choose the positive integer N as in [4, Lemma 7.4], corresponding to the length L and consider $n > N$.

Case-I If $\alpha^n \alpha^g$ is conjugate to $(\alpha^g)^n \alpha$ then there exists h in $\pi_1(F)$ such that $hA_{\alpha^n \alpha^g} = A_{(\alpha^g)^n \alpha}$. By [4, Lemma 7.4], $h\gamma_1(p, n) = \gamma_2(p, n)$ and hence their vertices coincide. Therefore by Remark 4, $\epsilon_1 = \epsilon_2$, a contradiction.

Case-II If $\alpha^n \alpha^g$ is conjugate to $((\alpha^g)^n \alpha)^{-1}$ then there exists h in $\pi_1(F)$ such that $hA_{\alpha^n \alpha^g} = A_{((\alpha^g)^n \alpha)^{-1}}$. Again by [4, Lemma 7.4], $h\gamma_1(p, n) = \gamma_2(p, n)$. Therefore from Figure 5 and from the construction of $\gamma_i(p, n)$, A_α coincides with some translate of $A_{\alpha^{-1}}$. This implies, α and α^{-1} are conjugate to each other in $\pi_1(F)$ which contradicts the fact that a hyperbolic element is never conjugate to its inverse in $PSL_2(\mathbb{R})$. Hence $\alpha^n \alpha^g$ is not conjugate to $((\alpha^g)^n \alpha)^{-1}$. \square

The proof of Lemma 2.1 follows at once from Lemma 3.1, Lemma 4.1 and Lemma 4.2.

Proof of Main theorem. Consider two intersection points P_1 and P_2 between α and β such that $\langle \alpha *_{P_1} \beta \rangle = \langle \alpha *_{P_2} \beta \rangle$. Therefore by Section 3, there exist $g, h \in \pi_1(F)$ such that $\langle \alpha \beta^g \rangle = \langle \alpha *_{P_1} \beta \rangle$, $\langle \alpha \beta^h \rangle = \langle \alpha *_{P_2} \beta \rangle$ and they are equal to each other. Using the same proof as in Lemma 4.1 we have, for any hyperbolic metric X on F , $l_X(\alpha^n \beta^g) = l_X(\alpha^n \beta^h)$ for all $n \in \mathbb{N}$. Now we use the same proof as the in Lemma 4.2. In **Case-I** if $\alpha^n \beta^g$ is conjugate to $\alpha^n \beta^h$ then from the proof we have $P_1 = P_2$ (because after a translation their vertices coincides) which gives a contradiction. The proof of **Case-II** is exactly the same. \square

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